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ON DOUBLE LIMITS AND ASYMPTOTIC SMALL SCALE YIELDING SOLUTIONS OF CRACK TIP FIELDS

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Abstract—The stress intensity factor emerged as a parameter quantifying the strength of crack tip stress or strain singularities in linear elastic materials but stress and strain intensity factors have also been used extensively to characterize crack tip stress and strain fields in elastic-plastic materials under small scale yielding conditions. A rigorous asymptotic small scale yielding solution must involve the passage to the limit as the two quantities r and R tend to zero, where r is the radius vector centred at the crack tip and R is the maximum crack tip plastic zone size. However, there is no unique solution, i.e. one which does not depend on the manner in which r and R tend to zero. In order to establish some contact between macroscopic continuum models and material microstructure an increasing number of fracture criteria stipulate the existence of a small finite characteristic length associated with the region neighbouring the crack tip, e.g. a process zone size r_z. In such cases characterization of the stress and strain states in the region bordering the process zone may be obtained from directed partial limit solutions corresponding to r, R tending to zero but the ratio ρ $(=r/R)$ tending to ρ_c $(=r_z/R_c)$ where R_c is the value of R at the initiation of crack extension. Quantities essentially similar to ρ_c have appeared in the literature from time to time and have also been encountered in connection with the crack separation energy rate G^{Δ} which was found to have a strong dependence on them.

NOTATION

a	abscissa of crack tip
E	modulus of elasticity
E'	$E/(1-\nu^2)$
$f_s(\theta)$	function of orientation angle θ
G	Griffith's energy release rate
G^{Δ}	crack separation energy rate
$h(N, \theta)$	function of hardening exponent N and orientation angle θ
J	path independent integral
K_I	Irwin's mode I stress intensity factor
$K^{\rho_c}_{\sigma}$	coefficient in directed partial limit solution for the stresses
$k^{\rho_c}_{\varepsilon}$	coefficient in directed partial limit solution for the strains
L, M, N	large numbers
N	power law hardening exponent
r, θ	polar co-ordinates with origin at the crack tip
r_n, θ_n	sequence of points near the crack tip
r_z	process zone size
R	maximum dimension of crack tip plastic zone
R_c	value of R at initiation of crack extension
R_m	sequence of crack tip plastic zone sizes
s	type of crack tip field singularity
S	sum of a double series of terms u_{ij}
S_{nm}	sequence of partial sums of a double series
u_{ij}	term of a double series
x, y	rectangular co-ordinates
x_n	x co-ordinates of sequence of points near the crack tip
α	engineer's offset in the definition of the flow stress
β	crack opening angle
δ	distance separating the crack surfaces at a point Δa from the crack tip
Δa	crack growth step $(=r_z)$
ε	equivalent strain
ε_E	elastic component of equivalent strain $(=\sigma/E)$
ε_p	plastic component of equivalent strain $(=\varepsilon-\varepsilon_E)$
ε_Y	σ_Y/E
$\varepsilon_{r\theta M}$	maximum shear strain when $r = r_z$, $(\theta = \theta_M)$
$\tilde{\varepsilon}_{r\theta}(N, \theta)$	function of hardening exponent N and orientation angle θ
ν	Poisson's ratio
ρ	r/R

ρ_c	r_{zi}/R_c
σ	equivalent stress
σ_α	stress component
$\sigma_{\alpha n}$	stress component at point r_n, θ_n
σ_{yy}	normal stress
σ_{yyn}	normal stress at point $(x_n, 0)$
σ_Y	yield stress in uniaxial tension
$\sigma_{\theta M}$	maximum tangential stress when $r = r_{zi}$, $(\theta = 0)$
$\tilde{\sigma}(N,\theta)$, $\tilde{\sigma}_\theta(N,\theta)$	functions of hardening exponent N and orientation angle θ.

1. INTRODUCTION

The singular field radial, tangential and shear stresses in the immediate neighbourhood of the tip of a crack in a linear elastic material under tensile loading normal to the crack are given by

$$\sigma_\alpha = K_I(2\pi r)^{-1/2}f_\alpha(\theta) \tag{1}$$

where r, θ are polar co-ordinates with the origin at the crack tip and the subscripts $\alpha = 1$, 2, 3 stand for r, θ and $r\theta$, respectively[1]. Directly ahead and in the plane of the crack the angular functions $f_\alpha(\theta)$ assume the values $f_r(0) = f_\theta(0) = 1$, $f_{r\theta}(0) = 0$. Here K_I is Irwin's mode I stress intensity factor which can be defined more precisely by stating that for any sequence of values $K_{In} = \sigma_{\alpha n}(2\pi r_n)^{1/2}f_\alpha(\theta)$, where $\sigma_{\alpha n}$ is a component of the stresses at the points r_n, θ_n with decreasing values of r_n such that $r_n \to 0$ as $n \to \infty$,

$$\lim_{n\to\infty} K_{In} = K_I. \tag{2}$$

The same value K_I is obtained irrespective of the approach path to the crack tip or of the stress component in the sequence. In particular, taking the approach path in the plane of the crack where θ vanishes and the stress component σ_θ, which is equal to the normal stress σ_{yy} referred to a rectangular co-ordinate system x, y with the abscissa in the plane of the crack and the ordinate normal to it,

$$K_{In} = \sigma_{yyn}(2\pi r_n)^{1/2}, \tag{3}$$

where here $r_n = x_n - a$, the crack tip being at the point $(a, 0)$.

Now consider a crack in an elastic–plastic material under small scale yielding (SSY) conditions. The SSY assumption implies that an asymptotic solution is sought for small values of the plastic zone size R. Thus we are led to consider also a sequence of decreasing plastic zones of size R_m, e.g. corresponding to increasing values of the yield stress, such that $R_m \to 0$ as $m \to \infty$. The stress intensity factor, if it exists, will be given by the double limit.

$$K_I' = \lim_{n,m\to\infty} K_{In,m}, \tag{4}$$

where K_{Inm} is defined as in eqn (3) but σ_{yyn} is the normal stress at the distance r_n ahead of the crack tip when the plastic zone size is R_m.

1.1. *Double series analogy*

It is perhaps illuminating to compare eqn (4) with one of the simplest cases of a quantity involving double limits, namely, the sum of an infinite double series of terms u_{ij} (Whittaker and Watson[34]). Thus if $S_{nm} = \Sigma_{i=1}^{n} \Sigma_{j=1}^{m} u_{ij}$ is a sequence of partial sums, where i, n and j, m can denote rows and columns, respectively, the infinite series is said to have the sum S if, given an arbitrarily small number η, there are numbers N and M such that for $n > N$ and $m > M$ or more simply for $n, m > L$ where L is the largest of N and M,

$$|S - S_{nm}| < \eta. \tag{5}$$

This definition of the sum of a double series implies that the sum is "path independent" in the sense that summations can take place in any combination of rows and columns, provided both n and m tend to infinity and the sum S will be the same for all of them. That this is not the case with K_I' in eqn (4) will be clear from the following examples.

Take a non-hardening elastic–plastic material and consider the limit defined in eqn (4) for the two cases (a) n, m, $(m/n) \to \infty$ and (b) n, m, $(n/m) \to \infty$. In the first case $(R_m/r_n) \to 0$ as $n, m \to \infty$ and K_I' assumes the value K_I given by eqn (2) for the linear–elastic material while in the second case $(r_n/R_m) \to 0$ as $n, m \to \infty$ and K_I' vanishes. Thus it cannot be said that the double limit defined by eqn (4) exists in the sense implied by inequality (5). We shall call the limits K_I obtained by assuming a given convergence path defined by $\rho_n = (r_n/R_m) \to \rho_c \geqslant 0$ as $n, m \to \infty$ "directed partial limits". Furthermore, it is to be expected that the *type* of the singularity, which we shall refer to as type $s(\rho_c)$ may change as ρ_c changes. The cases (a) and (b) discussed above then correspond to the directed partial limits K_I^∞ for s of type $r^{-1/2}$ and K_I^0 $(=0)$ for $s = 0$, i.e. for the degenerate case when the singularity disappears, respectively.

1.2. *Linear–elastic fracture mechanics (LEFM)*

In the traditional early application of LEFM to very brittle materials the existence of a small crack tip plastic zone or the occurrence of some kind of irreversible processes in the crack tip region is acknowledged but the size of this zone is assumed to be so small as to have insignificant effects on fracture processes. In fact the elastic parameters are used in assessing resistance to fracture, e.g. K_I as defined in eqn (2) for the linear–elastic material. This implies that the asymptotic solution adopted corresponds to the one connected with the directed partial limit K_I^∞ associated with a singularity of type $s = r^{-1/2}$ in this case and that the crack tip plastic zone is assumed to be completely embedded in, and small compared with the region where the stress and strain fields are thought to be most relevant to fracture processes.

1.3. *Elastic–plastic stress and strain intensity factors*

The now familiar Hutchinson[2, 3] and Rice–Rosengren[4] (HRR) singular solutions apply to idealized materials obeying power hardening law approximated by a Ramberg–Osgood type relation. If σ and ε are the equivalent stress and the equivalent strain, equal to the true stress and the natural strain in a uniaxial tension test, respectively, the "elastic" and "plastic" components of the strain can be defined as $\varepsilon_E = \sigma/E$ and $\varepsilon_p = \varepsilon - \varepsilon_E$, respectively, where E is the modulus of elasticity. Hutchinson uses the relation

$$\left(\frac{\sigma}{\sigma_Y}\right) = \alpha^{-N}\left(\frac{\varepsilon_p}{\varepsilon_Y}\right)^N, \tag{6}$$

where ε_Y is equal to σ_Y/E and the constant α is the "offset" in the engineer's definition of the flow stress σ_Y. Rice and Rosengren use $\sigma/\sigma_Y = (\varepsilon/\varepsilon_Y)^N$ and their material is incompressible. The normal range of values of the exponent N is given by $0 \leqslant N \leqslant 1$. At the lower end, when $N = 0$ the material is non-hardening and $\sigma \equiv \sigma_Y$ when $\varepsilon \geqslant \varepsilon_Y$. At the upper end, when $N = 1$ the response is linear given by $\sigma = [E/(1+\alpha)]\varepsilon$.

The HRR solution applies strictly to a non-linear elastic material or deformation theory plasticity, implying proportional loading. Since unloading is excluded the analysis on monotonically loaded static cracks cannot be extrapolated to growing cracks.

In contrast with the traditional LEFM application discussed earlier the SSY singular asymptotic HRR solution corresponds to the directed partial limit K_I^0 since the stress and strain fields under examination must be considered as being embedded entirely within the plastic region and the limiting value of the ratio $\rho(=r/R)$ equals 0 as the crack tip is approached. The stress and strain singularities are of types equal to $r^{-N/(1+N)}$ and $r^{-1/(1+N)}$, respectively. For positive values of N the stress is not bounded by the stress–strain law given by eqn (6) as the strain tends to infinity. Note however that in the case of a more realistic material which does not harden indefinitely but eventually saturates the stress singularity would not be present since sufficiently close to the crack tip the exponent N would vanish.

2. DISCUSSION

A crack tip in an infinite plane, sometimes referred to as the Inglis configuration, is useful for the study of crack tip fields when it is desirable to reduce to a minimum unwanted geometrical factors which are irrelevant to the study in question. When the material is linear–elastic, the only linear dimension provided by the configuration is the crack length $2a$, which however, is considered large compared with the region of interest dominated by the crack tip singular fields. But for an elastic–plastic material, the crack tip plastic zone size R is a suitable dimension with respect to which the radius r can be normalized giving non-dimensional $\rho = r/R$. This is often carried out, e.g. Hutchinson finds that the set of HRR solutions, corresponding to different values of the remotely applied stress, collapses after non-dimensionalization to one set of similar solutions, the normalized radius r_0 being equal to ρ times a constant.

Nevertheless, this continuum model is wanting for the proper representation of fracture phenomena, an important shortcoming being its total insensitivity to material microstructural properties. This is easily seen if one considers the hypothetical case of two macroscopically identical specimens made of materials with the same elastic–plastic properties, i.e. having the same values of E, ν, σ_Y and hardening properties, but with different grain sizes, inclusion spacings, etc. so that the values of the last two in the first specimen are only one half those in the second. One should not be surprised if the fracture responses of the two specimens differ.

2.1. *Fracture criteria incorporating a characteristic linear dimension*

It is now fairly generally accepted that events causing crack tip growth and fracture do not occur at a single point, namely the crack tip, but they occur in a small finite region in the vicinity of the crack tip. For this reason it would seem that a growing number of fracture criteria and models attempting to describe stable crack growth based on conventional elastic–plastic continuum mechanics, also stipulate some small characteristic length associated with the crack tip neighbourhood, thus providing some contact with microstructural features[5], e.g. a process zone size r_z[6, 7], a given distance from the crack tip where strains[8] or stresses[9] are assessed. Crack advance by discrete growth steps is stipulated or adopted in Wnuk's final stretch criterion[10–14], in criteria based on the crack opening displacement at a specified distance from the crack tip[15, 16] and in criteria involving crack separation energy rates[17, 18], the crack growth step Δa being identified in this case with the process zone size r_z. The process zone is believed to consist of a small region ahead of the crack tip which becomes unstable rather suddenly when one or more of a number of potential fracture micromechanisms[19, 20] are actuated, e.g. brittle cleavage fracture or ductile fracture associated with coalescence, void formation and void growth etc. The elastic–plastic constitutive relations which apply to the bulk of the material do not apply to the process zone. An exact analysis of the conditions for the prediction of the dominant micromechanism and the establishment of appropriate macro constitutive relations for the process zone present formidable tasks although some attempts towards these aims are made, e.g. articles in Ref. [21]. An explicit evaluation of the process zone size is not possible at this stage and r_z is believed to depend also on a multitude of microstructural features influencing the potential micromechanism of fracture such as the grain size, the distance between inclusions, etc. and possibly energy balance considerations [22, 23].

2.2. *The parameter ρ_c*

In addition to the crack tip plastic zone size which is known to have a marked influence on fracture behaviour a perhaps more important parameter is given by the relative sizes of the process zone and the crack tip plastic zone at initiation of crack extension, R_c, i.e. the ratio $\rho_c = r_z/R_c$ which provides some measure of the state of brittleness or ductility. Thus a large value of ρ_c, in the region of, or greater than, unity would indicate a material in a very brittle state while a low value of ρ_c indicates a ductile condition. For most steels used in engineering practice ρ_c is moderate or small. Quantities analogous to ρ_c often appear in the literature in somewhat disguised forms. For instance, in their analyses of stable crack growth Rice[5], and Rice and Sorenson[16] have pointed out conceptual difficulties in the description of a crack opening angle, stating that it can be defined unambiguously only for crack growth in rigid-plastic solids. A meaningful definition is provided by the angle subtended at the crack tip by points on the opening separating surfaces, situated at a distance behind the crack tip equal to the process zone size r_z. They further stipulate that r_z is equal to the crack-tip-opening displacement at the initiation of crack extension δ_{Ic}, a reasonable assumption for materials with sufficient ductility enabling them to sustain stable crack growth. Hence $r_z = \delta_{Ic} \approx 0.65 J_{Ic}/\sigma_Y$. Using the usual expression for R (eqn (8) appearing later), gives $\rho_c = B\sigma_Y/E$, where $B \approx 3.25$. The authors comment on the strong dependence of the resistance to stable crack growth, on the ratio σ_Y/E, i.e. or ρ_c.

A similar strong dependence on ρ_c (or analogous parameters) was also found in the case of the crack separation energy rate G^Δ, the crack growth step Δa being identified with r_z in this case, when it will be recalled, for a non-hardening material, $G^\Delta/J \to 0$ when $\rho_c \to 0$[18, 24].

3. A DIFFERENT INTERPRETATION OF THE HRR SOLUTION

With some inconsequential differences in notation the asymptotic structure given by Rice and Rosengren[4] in their eqn (21) is

$$\sigma \to \sigma_Y[Rh(N,\theta)/r]^{N/(1+N)} \tag{7a}$$

$$\varepsilon \to (\sigma_Y/E)[Rh(N,\theta)/r]^{1/(1+N)}, \tag{7b}$$

as $r \to 0$, where $h(N,\theta)$ assumes unit maximum value. Here R is identified with a maximum crack tip plastic zone size when only the singular field is taken into account. Hence R does not represent the actual crack tip plastic zone size which can also depend on biaxiality, i.e. the first non-singular term of the Williams[25] eigenfunction expansion. The load can be characterized by K_I or by Rice's path independent integral J, where for SSY, in a material which behaves linearly in the elastic range, $K_I^2 = E'J$, with $E' = E/(1-\nu^2)$ for plane strain conditions considered here. The yield stress and the maximum crack tip plastic zone are related by

$$R = A(K_I/\sigma_Y)^2 \quad \text{or} \quad \sigma_Y = (A/R)^{1/2}K_I, \tag{8}$$

where A is a number in the region of 0.2 and which takes the value $(2\pi)^{-1}$ when $N = 1$. Recalling that $\rho = r/R$ and substituting for σ_Y from (8), eqns (7) can be rewritten in the following form emphasizing the dependence of σ and ε on ρ and r,

$$\sigma \to (2\pi A)^{1/2}K_I h^{N/(1+N)}(N,\theta)\rho^{(1-N)/(2+2N)}(2\pi r)^{-1/2} \tag{9a}$$

$$\varepsilon \to (2\pi A)^{1/2}E^{-1}K_I h^{1/(1+N)}(N,\theta)\rho^{-(1-N)/(2+2N)}(2\pi r)^{-1/2}, \tag{9b}$$

as $r \to 0$. The SSY condition is satisfied when $\rho > 0$ since also $R(=r/\rho) \to 0$. Note that the solution is unique in the sense of being independent of the limiting value of ρ only when $N = 1$ corresponding to the linear solution. When $0 < N < 1$ uniqueness as r, $R \to 0$ is achieved by the further stipulation that ρ tends to some limiting value ρ_c as r, $R \to 0$. In the usual interpretation of the HRR solution $\rho_c = 0$, as was stated earlier, eqns (9) becoming meaningless in this case. However if it is intended to apply the asymptotic HRR solution with the aim of characterizing the behaviour over a zone which surrounds the region of finite strains and fracture processes wherein the validity of eqns (7) breaks down[26], the maximum tangential stress $\sigma_{\theta M}$, i.e. when $\theta = 0$ or the maximum shear strain $\varepsilon_{r\theta M}$ when $\theta = \theta_M$, at the boundary of the zone, where $r = r_z$ and $\rho = \rho_c$, may provide the required parameters:

$$\sigma_{\theta M} = K_{I\sigma}^{\rho}(2\pi r_z)^{-1/2} \tag{10a}$$

$$\varepsilon_{r\theta M} = K^{\rho}_{I\varepsilon}(2\pi r_z)^{-1/2}, \tag{10b}$$

where

$$K^{\rho}_{I\sigma} = (2\pi A)^{1/2}\tilde{\sigma}_\theta(N, \theta)\rho_c^{(1-N)/(2+2N)}K_I \tag{11a}$$

and

$$K^{\rho}_{I\varepsilon} = (2\pi A)^{1/2}E^{-1}\tilde{\varepsilon}_{r\theta}(N, \theta_M)\rho_c^{-(1-N)/(2+2N)}K_I. \tag{11b}$$

The non-dimensional functions $\tilde{\sigma}_\theta(N, \theta)$ and $\tilde{\varepsilon}_{r\theta}(N, \theta)$ of the orientation angle θ are the ones used by Hutchinson[3, p. 344] and are given for the values of $N = 1/3$ and $N = 1/13$, the normalization carried out by Hutchinson being such that the maximum value of the function $\tilde{\sigma}(N, \theta)$ associated with the equivalent stress is unity.

Note that $\tilde{\sigma}(N, \theta) = h^{N/(1+N)}(N, \theta)$.

Equations (10) cannot be interpreted as field equations in a simple manner since r_z and R (and therefore σ_Y) are not independent. Nevertheless the strengths of the singularities in the asymptotic solution (10) where r_z and R tend to zero and ρ tends to ρ_c, given by the directed partial limits $K^{\rho}_{I\sigma}$ and $K^{\rho}_{I\varepsilon}$ in eqns (11) may serve to characterize the states of stress and strain in the region immediately surrounding the process zone. When $N = 1$, $K^{\rho}_{I\sigma} = K_I$ as expected. For ρ and N both smaller than unity $(K_I/K^{\rho}_{I\sigma})$ increases as ρ_c decreases, i.e. in the direction of the greater ductility but the opposite is true of $(K_I/EK^{\rho}_{I\varepsilon})$. The non-hardening case when $N = 0$ corresponds to the perfect plasticity solution of Prandtl[27] and Hill[28] also discussed by Rice[29], revealing an absence of stress singularity but a shear strain singularity of type r^{-1} at the crack tip in the fan region of the Prandtl slip line field.

4. MATERIALS WITH TEMPERATURE DEPENDENT YIELD STRESSES

An example from data extracted from the literature may be used to illustrate the possibilities of using the partial limits $K^{\rho}_{I\sigma}$ or $K^{\rho}_{I\varepsilon}$ in a plastic criterion (without however the need to comment on the merits of the criteria). For instance, the fracture response of a high nitrogen mild steel at different cryogenic temperatures is described by Ritchie *et al.*[9] who use a criterion based on the attainment of a critical stress σ_f calculated (e.g. by using the HRR solution) at a point situated one or two grain sizes ahead of the crack tip in order to satisfy the "size scale" requirement. The best correlation with experimental results was obtained with the larger distance, namely 120 μm which can be equated to the process zone size r_z. The material obeys power-law hardening with an exponent $N = 0.1$. The critical stress $\sigma_f = 860$ MPa. All material properties with the exception of the yield stress are assumed to be independent of the temperature T. The yield stress v temperature curve shows a decrease in σ_Y as T increases. With T increasing and σ_Y decreasing, the analyses reveal that an increase of the applied load is needed to raise the stress to the required level for the satisfaction of the fracture criterion.

An analogous fracture criterion can be based on the attainment by $K^{\rho}_{I\sigma}$ of a critical value $K^{\rho}_{I\sigma c}$.

Thus, from eqns (8) and (11a),

$$K_{Ic} = C\sigma_Y^{-(1-N)/2N}, \tag{12}$$

where

$$C = [(2\pi A)^{-1/2}\tilde{\sigma}_\theta^{-1}(N, \theta)K^{\rho}_{I\sigma c}]^{(1+N)/2N}(A/r_z)^{(1-N)/4N}.$$

Using the values mentioned earlier for r_z, N and substituting the value of $\sigma_f(= 860$ MPa) in eqn (10a) we get $K^{\rho}_{I\sigma c} = 23.61$ MNm$^{-3/2}$. From Rice and Rosengren[4] we find $A = 0.22$ and from Hutchinson[2, 3] we estimate $\tilde{\sigma}_\theta(N, \theta) = 2.45$, yielding $C = 2.33 \times 10^{12}$. This result seems to be quite sensitive to the value of σ_f, e.g. for $\sigma_f = 830$ MPa, $C = 1.92 \times 10^{12}$.

Table 1. K_{Ic} at different temperatures using various fracture criteria

		Temperature T (°C)		
		−90	−80	−70
Yield stress σ_Y, MPa		269	250	237
K_Q experimental, $MNm^{-3/2}$		30.0	33.0	36.5
K_R corrected for plastic zone		34.0	40.0	46.5
K_{IC} ($K^{\rho}_{I\sigma c} = 23.6\ MNm^{-3/2}$)		27.1	37.7	48.0
	$\rho_c =$	0.054	0.024	0.133
K_{Ic} ($K^{\rho}_{I\sigma c} = 22.8\ MNm^{-3/2}$)		22.4	31.1	39.5
	$\rho_c =$	0.079	0.035	0.020
K_{Ic} ($E\varepsilon_{r\theta f} = 2409$ MPa)		23.6	22.8	22.3
	$\rho_c =$	0.071	0.066	0.062
K_{IC}($G^{\Delta}_c = 22.5\ MNm^{-3/2}$)		30.4	34.7	39.3
	$\rho_c =$	0.043	0.028	0.020

Table 1 gives K_{Ic}, calculated by using both values of C, at three different temperatures and the corresponding values of ρ_c are shown under those for K_{Ic}. The table also gives the experimental values K_Q obtained by Ritchie *et al.* However, the authors point out that SSY conditions did not prevail at temperatures above −95°C and corrected experimental values K_R were derived to account for the plastic zone when ASTM conditions were not met. These are also shown in Table 1.

In a strain criterion based on the attainment of a critical value for $K^{\rho}_{I\varepsilon}$ we have

$$K_{Ic} = D\sigma_Y^{(1-N)/2} \tag{13}$$

where

$$D = [(2\pi A)^{-1/2}\tilde{\varepsilon}_{r\sigma}^{-1}(N, \theta_M)EK^{\rho}_{I\varepsilon c}]^{(1+N)/2}(A/r_z)^{-(1-N)/4}.$$

Taking a value of $E\varepsilon_{r\theta f} = 2409$ MPa, where $\varepsilon_{r\theta f}$ is a critical strain at $r = r_z$ and estimating $\tilde{\varepsilon}_{r\sigma}(N, \sigma_M) = 0.81$ from Hutchinson (1978), we get $D = 1.90$. Calculated values of K_{Ic} using this criterion are also shown in Table 1. As expected they show a downward trend with increasing temperature. The quantity $K^{\rho}_{I\varepsilon}$ may be more relevant to strain dependent phenomena, e.g. fracture initiation or fatigue crack propagation. Finally, values of K_{Ic} based on a critical crack separation energy rate criterion, using the dependence of G^{Δ} on ρ_c, are also given. The data on Table 1 is reproduced visually in Fig. 1.

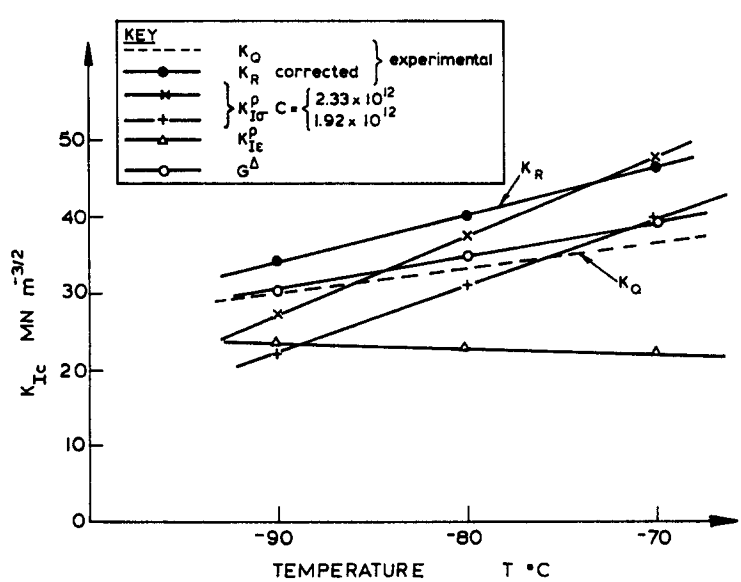

Fig. 1. K_{Ic} vs temperature for various fracture criteria.

4.1. *A note on load biaxiality*

Since the crack tip plastic zone size R' can be affected by the mode of load biaxiality [30, 31], etc., and if it can be assumed that r_z is independent of biaxiality, one is tempted to consider $\rho'_c = r_z/R'$ in order to take into account load biaxiality[33]. We note in passing that under SSY conditions J is not affected by load biaxiality when the material response is linear in the elastic range[32, 33].

5. CONCLUSIONS

1. The asymptotic small scale yielding solution for the singular crack tip stress and strain fields must involve passage to the limit as the two quantities r and R tend to zero, where r is the radius vector centred at the crack tip and R is the maximum crack tip plastic zone size. Strictly, a limiting solution does not exist, i.e. one which is independent of the manner in which r and R tend to zero.

2. Thus the directed partial limit solution when r, $R \to 0$ and $\rho \to \infty$, where $\rho = r/R$, corresponds to the elastic solution used traditionally in linear elastic fracture mechanics applications, while the solution corresponding to $r, R, \rho \to 0$ corresponds to the Hutchinson–Rice–Rosengren solution, r being completely embedded within the plastic zone.

3. Some contact with microstructure in continuum models is provided by the increasing number of fracture criteria which stipulate a characteristic length dimension in the crack tip region, e.g. a process zone size r_z, and the ratio $\rho_c = r_z/R_c$ is very relevant to fracture processes. The coefficients $K_{1\sigma}^{\rho}$ and $K_{1\varepsilon}^{\rho}$ of the term $(2\pi r_z)^{-1/2}$ appearing in the partial solution corresponding to r, $R \to 0$ and $\rho \to \rho_c$ may perhaps be used to characterize the states of stress or strain in the region bordering the process zone.

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